A calculus on Lévy exponents and selfdecomposability on Banach spaces*

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ABSTRACT. In infinite dimensional Banach spaces there is no complete characterization of the Lévy exponents of infinitely divisible probability measures. Here we propose *a calculus on Lévy exponents* that is derived from some random integrals. As a consequence we prove that *each* selfdecomposable measure can by factorized as another selfdecomposable measure and its background driving measure that is s-selfdecomposable. This complements a result from the paper of Iksanov-Jurek-Schreiber in the Annals of Probability **32**, 2004.

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Abbreviated title: *A calculus on Lévy exponents*

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1. Introduction. Recall that a Borel probability measures μ, on a real separable Banach space E, is called *infinitely divisible* if for each natural number n there exists a probability measure μ_n such that $\mu_n^{*n} = \mu$; the class of all infinitely divisible measures will be denoted by ID. It is well-know that their Fourier transforms (*the Lévy-Khintchine formulas*) can be written as follows

$$\hat{\mu}(y) = e^{\Phi(y)},\ y \in E', \quad \text{and the exponents } \Phi \text{ are of the form}$$

$$\Phi(y) = i < y, a > -\frac{1}{2} < y, Ry > + \int_{E\setminus\{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)]M(dx), \tag{1}$$

where E' denote the dual Banach space, $< .,. >$ is an appropriate bilinear form between E' and E, a is a *shift vector*, R is a *covariance operator* corresponding to the Gaussian part of μ and M is a *Lévy spectral measure.* There is a one-to-one corresponds between $\mu \in ID$ and the triples $[a, R, M]$ in its Lévy-Khintchine formula (1); cf. Araujo-Giné (1980), Chapter 3, Section 6, p. 136. The function $\Phi(y)$ from (1) is called the *Lévy exponent* of μ.

REMARK 1. (a) If E is a Hilbert space then Lévy spectral measures M are completely characterized by the integrability condition $\int_E (1 \wedge ||x||^2)M(dx) < \infty$ and Gaussian covariance operators R coincide with the positive trace-class operators ; cf. Parthasarathy (1967), Chapter VI, Theorem 4.10.

(b) When E is *an Euclidean space* then Lévy exponents are completely characterized as continuous negative-definite functions; cf. Cuppens (1975) and Schoenberg's Theorem on p. 80.

Finally, a *Lévy process* $Y(t), t \geq 0$, means a continuous in probability process with stationary and independent increments and $Y(0) = 0$. Without loss of generality we may and do assume that it has paths in the Skorochod space $D_E[0, \infty)$ of E-valued *cadlag functions* (i.e., right continuous with left hand limits). There is a one-to-one correspondence between the class ID and the class of Lévy processes.

The cadlag paths of a process Y allows us define *random integrals* of the form $\int_{(a,b]} h(s)Y(r(ds))$ via the formal formula of integration by parts.

Namely,

$$\int_{(a,b]} h(s)Y(r(ds)) :=$$

$$h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(s))dh(s), \tag{2}$$

where h is a real valued function of bounded variation and r is a monotone and right-continuous function. Furthermore, we have

$$\widehat{\mathcal{L}\Big(\int_{(a,b]} h(s)Y(r(ds))\Big)}(y) = \exp \int_{(a,b]} \log \widehat{\mathcal{L}(Y(1))}(h(s)y)dr(s), \tag{3}$$

where $\mathcal{L}(.)$ denotes the probability distribution and $\hat{\mu}(.)$ denotes the Fourier transform of a measure μ; cf. Jurek-Vervaat (1983) or Jurek (1985) or Jurek-Mason (1993), Section 3.6, p. 116.

2. A calculus on Lévy exponents. Let $\mathcal{E}$ denotes the totality of all functions $\Phi : E' \to \mathbb{C}$ appearing as the exponent in the Lévy-Khintchine formula (1). Hence we have that

$$\mathcal{E} + \mathcal{E} \subset \mathcal{E}, \quad \lambda \cdot \mathcal{E} \subset \mathcal{E}, \ \text{ for all postive } \ \lambda, \tag{4}$$

which means that $\mathcal{E}$ forms a cone in the space of all complex valued functions defined on E'. Furthermore, if $\Phi \in \mathcal{E}$ then all dilations $\Phi(a\cdot) \in \mathcal{E}$. These follow from the fact that infinite divisibility is preserved under convolution and under (convolution) powers to positive real numbers.
Here we consider two integral operators acting on $\mathcal{E}$ or its part. Namely,

$$\begin{aligned} &\mathcal{J} : \mathcal{E} \to \mathcal{E}, \ \ (\mathcal{J}\Phi)(y) := \int_0^1 \Phi(sy)ds, \ \ y \in E'; \\ &\mathcal{I} : \mathcal{E}_{\log} \to \mathcal{E}, \qquad (\mathcal{I}\Phi)(y) := \int_0^1 \Phi(sy)s^{-1}ds, \ \ y \in E'. \end{aligned} \tag{5}$$

Note that $\mathcal{J}$ is well defined on all of$\mathcal{E}$, since by (3), $\mathcal{J}\Phi$ is the Lévy exponent of the well-defined integral $\int_{(0,1]} tdY(t)$, where $Y(1)$ has the Lévy exponent Φ; cf. Jurek (1985) or (2004). On the other hand, $\mathcal{I}$ is only defined on $\mathcal{E}_{\log}$, which corresponds to infinitely divisible measures with finite

logarithmic moments, since $\mathcal{I}\Phi$ is the Lévy exponent of the random integral $\int_{(0,1]} t dY(-\ln t) = \int_{(0,\infty)} e^{-s} dY(s)$, where Φ is the Lévy exponent of $Y(1)$ that has finite logarithmic moment; cf. Jurek-Vervaat (1983).

Here are the main algebraic properties of the mappings $\mathcal{J}$ and $\mathcal{I}$.

LEMMA 1. *The operators $\mathcal{I}$ and $\mathcal{J}$ acting on appropriate domains (Lévy exponents) have the following basic properties:*

(a) $\mathcal{I}, \mathcal{J}$ *are additive and positive homogeneous operators;*

(b) $\mathcal{I}, \mathcal{J}$ *commute under the composition and* $\mathcal{J}(\mathcal{I}(\Phi)) = (\mathcal{I} - \mathcal{J})\Phi$.

Other equivalent forms of that last property are:
$\mathcal{J}(I + \mathcal{I}) = \mathcal{I}$; $\mathcal{I}(I - \mathcal{J}) = \mathcal{J}$; $(I - \mathcal{J})(I + \mathcal{I}) = I$.

Proof. Part (a) follows from the fact that $\mathcal{E}$ forms a cone. For part (b) note that

$$(\mathcal{J}(\mathcal{I}(\Phi)))(y) = \int_0^1 (\mathcal{I}(\Phi))(ty)\, dt = \int_0^1 \int_0^1 \Phi(sty) s^{-1} ds dt =$$
$$\int_0^1 \int_0^t \Phi(ry) r^{-1} dr dt = \int_0^1 \int_r^1 \Phi(ry) dt\, r^{-1} dr =$$
$$\int_0^1 \Phi(ry) r^{-1} dr - \int_0^1 \Phi(ry) dr = \mathcal{I}\Phi(y) - \mathcal{J}\Phi(y) = (\mathcal{I} - \mathcal{J})\Phi(y),$$

which proves the equality in (b). Note that from the above (the first line of the above argument) we infer also that that operators $\mathcal{I}$ and $\mathcal{J}$ commute which completes the argument. □

LEMMA 2. *The operators $\mathcal{I}$ and $\mathcal{J}$, defined by (5), have the following additional properties:*

(a) $\mathcal{J} : \mathcal{E}_{\log} \to \mathcal{E}_{\log}$ *and* $\mathcal{I} : \mathcal{E}_{(\log)^2} \to \mathcal{E}_{\log}$,

(b) *If* $(I - \mathcal{J})\Phi \in \mathcal{E}$ *then the corresponding infinitely divisible measure $\tilde{\mu}$ with the Lévy exponent $(I - \mathcal{J})\Phi(y)$, $y \in E'$, has finite logarithmic moment.*

(c) $(I-\mathcal{J})\Phi + \mathcal{I}(I-\mathcal{J})\Phi \;=\; (I-\mathcal{J})\Phi + \mathcal{J}\Phi \;=\; \Phi \quad$ *for all* $\;\Phi \in \mathcal{E}.$

Proof. (a) Since the function $E \ni x \to \log(1+||x||)$ is sub-additive, for an infinitely divisible probability measure $\mu = [a, R, M]$ we have

$$\int_E \log(1+||x||)\mu(dx) < \infty \;\text{ iff }\; \int_{\{||x||>1\}} \log(1+||x||)M(dx) < \infty$$
$$\text{iff} \int_{\{||x||>1\}} \log||x||M(dx) < \infty; \tag{6}$$

cf. Jurek and Mason (1993), Proposition 1.8.13. Furthermore, if M is the spectral Lévy measure appearing in the Lévy exponent Φ then $\mathcal{J}\Phi$ has Lévy spectral measure $\mathcal{J}M$ (we keep that potentially conflicting notation), where

$$(\mathcal{J}M)(A) := \int_{(0,1)} M(t^{-1}A)dt = \int_{(0,1)} \int_E 1_A(tx)M(dx)dt, \tag{7}$$

for all Borel subsets A of $E \setminus \{0\}$. Hence

$$\int_{\{||x||>1\}} \log||x||(\mathcal{J}M)(dx) = \int_{(0,1)} \int_E 1_{\{||x||>1\}}(tx)\log(t||x||)M(dx)dt$$
$$= \int_{(0,1)} \int_{\{||x||>t^{-1}\}} \log(t||x||)M(dx)dt = \int_{\{||x||>1\}} \int_{||x||^{-1}}^{1} \log(t||x||)dt\, M(dx)$$
$$= \int_{\{||x||>1\}} ||x||^{-1} \int_1^{||x||} \log w\, dw\, M(dx)$$
$$= \int_{\{||x||>1\}} ||x||^{-1}[||x||\log||x|| - ||x|| + 1]M(dx)$$
$$= \int_{\{||x||>1\}} \log||x||M(dx) - \int_{\{||x||>1\}} [1-||x||^{-1}]M(dx).$$

Since the last integral is always finite as we integrate a bounded function with respect to a finite measure, we get the first part of (a). For the second one, let us note that

$$\int_{\{||x||>1\}} \log||x||(\mathcal{I}M)(dx) = \int_0^\infty \int_{\{||x||>1\}} \log||x||M(e^t dx)dt$$
$$= 1/2 \int_{\{||x||>1\}} \log^2||x||M(dx),$$

where $\mathcal{I}M$ is the Lévy spectral measure corresponding to the Lévy exponent $\mathcal{I}\Phi$.

For the part (b), note that the assumption made there implies that the measure

$$\widetilde{M}(A) := M(A) - \int_{(0,1)} M(t^{-1}A)dt \geq 0, \quad \text{for all Borel sets } A \subset E \setminus \{0\}, \quad (8)$$

is the Lévy spectral measure of some $\tilde{\mu}$. [Note that there is no restriction on the Gaussian part.] In fact, if $\widetilde{M}$ is a nonnegative measure then it is necessarily a Lévy spectral measure because $0 \leq \widetilde{M} \leq M$ and M is Lévy spectral measure; comp. Arujo-Giné (1980), Chapter 3, Theorem 4.7 , p. 119. To establish the logarithmic moment of $\tilde{\mu}$ we argue as follows. Observe

that for any constant $k > 1$ we have

$$
\begin{aligned}
0 \le \int_{(\{1<||x||\le k\}} & \log ||x|| \widetilde{M}(dx) = \\
& \int_{\{1<||x||\le k\}} \log ||x|| M(dx) - \int_{(0,1)} \int_{\{1<||x||\le k\}} \log ||x|| M(t^{-1}dx)dt = \\
& \int_{\{1<||x||\le k\}} \log ||x|| M(dx) - \int_{(0,1)} \int_{\{t^{-1}<||x||\le kt^{-1}\}} \log(t||x||) \, dM(dx)dt = \\
& \int_{\{1<||x||\le k\}} \log ||x|| M(dx) - \int_{\{1<||x||\le k\}} \int_{||x||^{-1}}^{1} \log(t||x||)dt \, M(dx) \\
& \qquad - \int_{\{k<||x||\}} \int_{||x||^{-1}}^{k||x||^{-1}} \log(t||x||)dt \, M(dx) = \\
& \int_{\{1<||x||\le k\}} \log ||x|| M(dx) - \int_{\{1<||x||\le k\}} ||x||^{-1} \int_{1}^{||x||} \log(w) \, dw \, M(dx) \\
& \qquad - \int_{\{k<||x||\}} ||x||^{-1} \int_{1}^{k} \log(w) \, dw \, M(dx) = \\
& \int_{\{1<||x||\le k\}} \log ||x|| M(dx) - \int_{\{1<||x||\le k\}} ||x||^{-1}(||x|| \log ||x|| - ||x|| + 1) M(dx) \\
& \qquad - (k \log k - k + 1) \int_{\{||x||>k\}} ||x||^{-1} M(dx) = \\
& \int_{\{1<||x||\le k\}} (1 - ||x||^{-1}) M(dx) - (k \log k - k + 1) \int_{\{||x||>k\}} ||x||^{-1} M(dx) \\
& \qquad \le M(||x|| > 1) < \infty,
\end{aligned}
$$

and consequently $\int_{(||x||>1)} \log ||x|| \widetilde{M}(dx < \infty$. This with property (6), completes the proof of the part (b).

Finally, since $(I - \mathcal{J})\Phi$ is in a domain of definition of the operator $\mathcal{I}$, so the part (c) is a consequence of Lemma 1(e) and (d). Thus the proof is complete. □

3. Factorizations of selfdecomposable distributions. The classes of limit laws $\mathcal{U}$ and L are obtained by non-linear shrinking transformations and linear transformations (multiplications by scalars), respectively; cf. Jurek (1985) and references there. However, there are many (unexpected) relations

between $\mathcal{U}$ and L as was already proved in Jurek (1985) and more recently in Iksanov-Jurek-Schreiber (2004). Furthermore, more recently selfdecomposable distributions are used in modelling real phenomena, in particular in mathematical finance; for instance cf. Bingham (2006), Carr-Geman-Madan-Yor (2005) or Eberlein-Keller (1995). This motivates further studies on factorizations and other relations between the classes $\mathcal{U}$ and L, like those in Theorems 1 and 2, below.

In this section we will apply the operators $\mathcal{I}$ and $\mathcal{J}$ to Lévy exponents of selfdecomposable (the class L) and s-selfdecomposable (the class $\mathcal{U}$) probability measures. For the convenience of the readers recall here that

$$\begin{aligned} \mu \in L \ \text{ iff } \ &\forall(t>0)\exists\nu_t \ \ \mu = T_{e^{-t}}\,\mu * \nu_t \\ &\text{iff } \ \mu = \mathcal{L}\Big(\int_{(0,\infty)} e^{-t} dY(t)\Big); \ \ \mathcal{L}(Y(1)) \in ID_{\log}, \\ \mu \in \mathcal{U} \ \text{ iff } \ &\mu = \mathcal{L}\Big(\int_{(0,1]} t\, dY(t)\Big), \ \ \mathcal{L}(Y(1)) \in ID. \end{aligned} \tag{9}$$

Meaures from the class $\mathcal{U}$ are called *s-selfdecomposable*; cf Jurek (1985), (2004). The corresponding Fourier transforms of measures from L and $\mathcal{U}$ easily follow from (2) and (3); cf. Jurek-Vervaat (1983) or the above references.

LEMMA 3. *If μ is a selfdecomposable probability measure on a Banach space E with characteristic function $\hat{\mu}(y) = \exp[\Phi(y)]\, y \in E'$, then*

$$\widetilde{\Phi}(y) := \Phi(y) - \int_{(0,1)} \Phi(sy) ds = (I - \mathcal{J})\Phi(y), \ y \in E',$$

is a Lévy exponent corresponding to an infinitely divisible probability measure with finite logarithmic moment.

Equivalently, if M is the Lévy spectral measure of a selfdecomposable μ then the measure $\widetilde{M}$ given by

$$\widetilde{M}(A) := M(A) - \int_0^1 M(t^{-1}A)dt, \quad A \subset E \setminus \{0\},$$

is a Lévy spectral measure on E that additionally integrates the logarithmic function on the complement of any neighborhood of zero.

Proof. If $\mu = [a, R, M]$ is selfdecomposable (or in other words a class L distribution) then we infer that

$$M(A) - M(e^t A) \geq 0, \quad \text{for all} \;\; t > 0 \text{ and Borel } \;\; A \subset E \setminus \{0\},$$

and that there is no restriction on the remaining two parameters (the shift vector and the Gaussian covariance operator) in the Lévy-Khintchine formula (1). Multiplying both sides by e^{-t} and then integrating over the positive half-line we conclude that $\widetilde{M}$, given by (8), is a non-negative measure. Since $\widetilde{M} \leq M$ and M is a Lévy spectral measure, so is $\widetilde{M}$; comp. Theorem 4.7 in Chapter 3 of Araujo-Giné (1980). Finally, our Lemma 2(b) gives the finiteness of the logarithmic moment. Thus the proof is complete. $\square$

THEOREM 1. *For each selfdecomposable probability measure μ, on a Banach space E, there exists a unique s-selfdecomposable probability measure $\tilde{\mu}$ with finite logarithmic moment such that*

$$\mu = \tilde{\mu} * \mathcal{I}(\tilde{\mu}) \quad \textit{and} \quad \mathcal{J}(\mu) = \mathcal{I}(\tilde{\mu}) \;. \tag{10}$$

In fact, if $\hat{\mu}(y) = \exp[\Phi(y)]$ then $(\tilde{\mu})\hat{}(y) = \exp[\Phi(y) - \int_{(0,1)} \Phi(ty)dt]$, $y \in E'$.

In other words, if Φ is the Lévy exponent of a selfdecomposable probability measure then $(I - \mathcal{J})\Phi$ is the Lévy exponent of an s-selfdecomposable measure with the finite logarithmic moment and

$$\Phi = (I - \mathcal{J})\Phi + \mathcal{I}(I - \mathcal{J})\Phi = (I - \mathcal{J})\Phi + \mathcal{J}\Phi. \tag{11}$$

Proof. Let $\hat{\mu}(y) = \exp[\Phi(y)] \in L$. From the factorization in (9) (the first line) we infer that $\Phi_t(y) := \Phi(y) - \Phi(e^{-t}y)$ are Lévy exponents as well. Hence,

$$\widetilde{\Phi}(y) := \int_{(0,\infty)} \Phi_t(ty)e^{-t}dt = \Phi(y) - \int_{(0,\infty)} \Phi(e^{-t}y)e^{-t}dt = ((I - \mathcal{J})\Phi)(y)$$

is a Lévy exponent as well, because of Lemma 3. Again by Lemma 3 (or Lemma 2 b)), a probability measure $\tilde{\mu}$ defined by the Fourier transform $(\tilde{\mu})\hat{}(y) = \exp(I - \mathcal{J})\Phi(y)$ has logarithmic moment. Consequently, $\mathcal{I}(\tilde{\mu})$ is a well defined probability measure whose Lévy exponent is equal to $\mathcal{I}(I - \mathcal{J})\Phi$. Finally, Lemmas 1(b) and 2(c ???) give the factorization (10).

Since $\mathcal{I}(\tilde{\mu}) \in L$ has the property that $\tilde{\mu} * \mathcal{I}(\tilde{\mu})$ is again in L, therefore Theorem 1 from Iksanov-Jurek-Schreiber(2004) gives that $\tilde{\mu} \in \mathcal{U}$, i.e., it is a s-selfdecomposable probability distribution.

To see the second equality in (11) one should observe that it is equivalent to equality $\mathcal{J}\Phi = \mathcal{I}(I - \mathcal{J})\Phi$ that indeed holds true in view of Lemma 1(d).

Suppose there exists another factorization of the form $\mu = \rho * \mathcal{I}(\rho)$ and let $\Xi(y)$ be the Lévy exponent of ρ. Then we get that $\Phi(y) = \Xi(y) + (\mathcal{I}\,\Xi)(y) = (I + \mathcal{I})\,\Xi(y)$. Hence, applying to both sides $\mathcal{I} - \mathcal{J}$ we conclude that

$$(I - \mathcal{J})\Phi = ((I - \mathcal{J})(I + \mathcal{I}))\,\Xi = \Xi,$$

where the last equality is from Lemma 1(b). This proves the uniqueness of $\tilde{\mu}$ in the representation (10) and thus the proof of Theorem 1 is completed. □

REMARK 2. The factorization (10), in Theorem 1, can be also derived from previous papers as follows:

for each selfdecomposable (or class L) μ there exists a unique $\rho \in ID_{\log}$ such that $\mu = \mathcal{I}(\rho)$; Jurek-Vervaat (1983). Since $\tilde{\mu} := \mathcal{J}(\rho)$ is s-selfdecomposable (class) $\mathcal{U}$) with logarithmic moment (cf. Jurek (1983)) therefore, $\mathcal{I}(\tilde{\mu}) * \tilde{\mu} \in L$ in view of Iksanov-Jurek-Schreiber (2004). Finally, again by Jurek (1985), $\mathcal{I}(\tilde{\mu}) * \tilde{\mu} = \mathcal{J}(\mathcal{I}(\rho) * \rho) = \mathcal{I}(\rho) = \mu$, which gives the decomposition.

However, the present proof is less involved, more straightforward and moreover the result and the proof of finiteness of the logarithmic moment in Lemma 2 (b) is completely new. Last but not least, the "calculus" on Lévy exponents, introduced in this note, is of an interest in itself.

REMARK 3. In the case of Euclidean space $\mathbb{R}^d$, using Schoenberg's Theorem, one gets immediately that $\widetilde{\Phi}$ is a Lévy exponent; cf. Cuppens (1975), pp. 80-82.

Following Iksanov, Jurek and Schreiber (2004), p. 1360, we will say that a selfdecomposable probability measure μ has *the factorization property* if $\mu * \mathcal{I}^{-1}(\mu)$ is selfdecomposable as well. In other words, a class L probability measure convolved with its background driving probability distribution is again class L distribution. As in Iksanov-Jurek-Schreiber (2004), Proposition 1, if L^f denotes the set of all class L distribution with the factorization property then

$$L^f = \mathcal{I}(\mathcal{J}(ID_{\log})) = \mathcal{J}(\mathcal{I}(ID_{\log})) = \mathcal{J}(L) \text{ and } L^f \subset L \subset \mathcal{U}, \qquad (12)$$

COROLLARY 1. *Each selfdecomposable μ admits a factorization $\mu = \nu_1 * \nu_2$, where ν_1 is an s-selfdecomposable measure (i.e., $\nu_1 \in \mathcal{U}$) and ν_2 is a selfdecomposable one with the factorization property (i.e., $\nu_2 \in L^f$). That is, besides the inclusion $L^f \subset L \subset \mathcal{U}$ we also have that $L \subset L^f * \mathcal{U}$.*

Proof. Because of (10), $\nu_1 := \tilde{\mu}$ is an s-selfdecomposable measure. Furthermore, $\nu_2 := \mathcal{I}(\tilde{\mu}) \in L$ has the factorization property, i.e., $\nu_2 \in L^f$, which completes the proof. $\square$

EXAMPLES. 1) Let Σ_p be a symmetric stable distribution on a Banach space E, with the exponent p. Then its Lévy exponent, Φ_p, is equal to $\Phi_p(y) = -\int_S | < y, x > |^p \, m(dx)$, where m is a finite Borel measure on the unit sphere S of E; cf. Samorodnitsky and Taqqu (1994). Hence $(I - \mathcal{J})\Phi_p(y) = p/(p+1)\Phi_p(y)$, which means that in Corollary 1, both ν_1 and ν_2 are stable with the exponent p and measures $m_1 := (p/(p+1))m$ and $m_2 := (1/(p+1))m$, respectively.

2) Let η denotes the Laplace (double exponential) distribution on real line $\mathbb{R}$; cf. Jurek-Yor (2004). Then its Lévy exponent Φ_η is equal to $\Phi_\eta(t) := -\log(1+t^2)$, $t \in \mathbb{R}$. Consequently, $(I - \mathcal{J})\Phi_\eta(t) = 2(\arctan t - t)t^{-1}$ is the Lévy exponent of the class $\mathcal{U}$ probability measure ν_1 from Corollary 1, and $(2t - \arctan t - t \, \log(1+t^2))t^{-1}$ is the Lévy exponent of the class L^f measure ν_2 from Corollary 1.

Before we formulate the next result we need to recall that, by (9), the class $\mathcal{U}$ is defined here as $\mathcal{U} = \mathcal{J}(ID)$. Consequently, by iteration argument we can define

$$\mathcal{U}^{<1>} := \mathcal{U}, \;\; \mathcal{U}^{<k+1>} := \mathcal{J}(\mathcal{U}^{<k>}) = \mathcal{J}^{k+1}(ID), \;\; k = 1, 2, ...; \qquad (13)$$

cf. Jurek (2004) for other characterization of classes $\mathcal{U}^{<k>}$. Elements from the semigropus $\mathcal{U}^{<k>}$ are called *k-times s-selfdecomposable probability measures.*

THEOREM 2. *Let n be any natural number and μ be a selfdecopmosable probability measure. Then there exist k-times s-selfdecomposable probability measures $\tilde{\mu}_k$, for $k = 1, 2, ..., n$, such that*

$$\mu = \tilde{\mu}_1 * \tilde{\mu}_2 * ... * \tilde{\mu}_n * \mathcal{I}(\tilde{\mu}_n), \;\; \mathcal{J}^k(\mu) = \mathcal{I}(\tilde{\mu}_k), \quad k = 1, 2, ..., n. \qquad (14)$$

In fact, if Φ is the exponent of μ then $\tilde{\mu}_k$ has the exponent $\mathcal{I}^{k-1}(I - \mathcal{J})^k\Phi = (I - \mathcal{J})\mathcal{J}^{k-1}\Phi$ and

$$\begin{aligned}\Phi = (I-\mathcal{J})\Phi + (I-\mathcal{J})\mathcal{J}\Phi + ... + (I-\mathcal{J})\mathcal{J}^{k-1}\Phi + ... + (I-\mathcal{J})\mathcal{J}^{n-1}\Phi + \mathcal{J}^n\Phi \\ = (I - \mathcal{J}^n)\Phi + \mathcal{J}^n\Phi. \qquad (15)\end{aligned}$$

Proof. For $n = 1$ the factorization (14) and the formula (15) are true by Theorem 1, with $\tilde{\mu}_1 := \tilde{\mu}$. Suppose our claim (14) is true for n. Since $\rho := \mathcal{I}(\tilde{\mu}_n)$ is selfdecomposable, applying to it Theorem 1, we have that $\rho = \tilde{\rho} * \mathcal{I}(\tilde{\rho})$, where $\tilde{\rho}$ has the Lévy exponent $(I - \mathcal{J})\mathcal{J}^n\Phi = \mathcal{J}^n(I - \mathcal{J})\Phi$ and thus it corresponds to $(n+1)$-times s-selfdecomposable probability because, by Theorem 1, $(I - \mathcal{J})\Phi$ is already s-selfdecomposable and then we apply n times the operator $\mathcal{J}$; compare the definition (13). Thus the factorization (14) holds for $n+1$, which completes the proof of the first part of the theorem. Similarly, applying inductively decomposition (11), from Theorem 1 and observing from Lemma 1(b) that we will get the formula (14). Thus the proof is complete. $\square$

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